On the Liouville-Arnold integrable flows related with quantum algebras and their Poissonian representations

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Abstract. Based on the structure of Casimir elements associated with general Hopf algebras there are constructed Liouville-Arnold integrable flows related with naturally induced Poisson structures on arbitrary co-algebra and their deformations. Some interesting special cases including the oscillatory Heisenberg-Weil algebra related co-algebra structures and adjoint with them integrable Hamiltonian systems are considered.

1 Hopf algebras and co-algebras: main definitions

Consider a Hopf algebra $\mathcal{A}$ over $\mathbb{C}$ endowed with two special homomorphisms called coproduct $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ and counit $\varepsilon : \mathcal{A} \to \mathbb{C}$, as well an antihomomorphism (antipode) $\nu : \mathcal{A} \to \mathcal{A}$, such that for any $a \in \mathcal{A}$

$$\begin{aligned} (id \otimes \Delta)\Delta(a) &= (\Delta \otimes id)\Delta(a), \\ (id \otimes \varepsilon)\Delta(a) &= (\varepsilon \otimes id)\Delta(a) = a, \\ m((id \otimes \nu)\Delta(a)) &= m((\nu \otimes id)\Delta(a)) = \varepsilon(a)I, \end{aligned} \tag{1.1}$$

where $m : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ is the usual multiplication mapping, that is for any $a, b \in \mathcal{A}$ $m(a \otimes b) = ab$. The conditions (1.1) were introduced by Hopf [1] in a cohomological context. Since most of the Hopf algebras properties depend on the coproduct operation $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ and related with it Casimir elements, below we shall dwell mainly on the objects called co-algebras endowed with this coproduct.

The most interesting examples of co-algebras are provided by the universal enveloping algebras $U(\mathcal{G})$ of Lie algebras $\mathcal{G}$. If, for instance, a Lie algebra $\mathcal{G}$ possesses generators $X_i \in \mathcal{G}$, $i = \overline{1, n}$, $n = \dim \mathcal{G}$, the corresponding enveloping

algebra $U(\mathcal{G})$ can be naturally endowed with a Hopf algebra structure by defining

$$\begin{aligned} \Delta(X_i) &= I\otimes X_i + X_i\otimes I,\ \Delta(I)=I\otimes I, \\ \varepsilon(X_i) &= -X_i, \qquad\qquad \nu(I)=-I. \end{aligned} \tag{1.2}$$

These mappings acting only on the generators of $\mathcal{G}$ are straightforwardly extended to any monomial in $U(\mathcal{G})$ by means of the homomorphism condition $\Delta(XY)=\Delta(X)\Delta(Y)$ for any $X,Y\in\mathcal{G}\subset U(\mathcal{G})$. In general< an element $Y\in U(\mathcal{G})$ of a Hopf algebra such that $\Delta(Y)=I\otimes Y+Y\otimes I$ is called primitive, and the known Friedrichs theorem [2] ensures, that in $U(\mathcal{G})$ the only primitive elements are exactly generators $X_i\in\mathcal{G},\ \ i=\overline{1,n}$.

On the other handside, the homomorphism condition for the coproduct $\Delta:\mathcal{A}\to\mathcal{A}\otimes\mathcal{A}$ implies the compatibility of the coproduct with the Lie algebra commutator structure:

$$[\Delta(X_i),\Delta(X_j)]_{\mathcal{A}\otimes\mathcal{A}}=\Delta([X_i,X_j]_{\mathcal{A}}) \tag{1.3}$$

for any $X_i,X_j\in\mathcal{G},\ \ i,j=\overline{1,n}$. Since the Drinfeld report [3] the co-algebras defined above are also often called "quantum" groups due to their importance [4] in studying many two-dimensional quantum models of modern field theory and statistical physics.

It was also observed (see for instance [4]), that the standard co-algebra structure (1.2) of the universal enveloping algebra $U(\mathcal{G})$ can be nontrivially extended making use of some its infinitesimal deformations< saving the co-associativity (1.3) of the deformed coproduct $\Delta: U_z(\mathcal{G})\to U_z(\mathcal{G})\otimes U_z(\mathcal{G})$ with $U_z(\mathcal{G})$ being the corresponding universal enveloping algebra deformation by means of a parameter $z\in\mathbb{C}$, such that $\lim\limits_{z\to 0}U_z(\mathcal{G})=U(\mathcal{G})$ subject to some natural topology on $U_z(\mathcal{G})$.

2 Casimir elements and their special properties

Take any Casimir element $C\in U_z(\mathcal{G})$, that is an element satisfying the condition $[C,U_z(\mathcal{G})]=0$, and consider the action on it of the coproduct mapping Δ :

$$\Delta(C)=C(\{\Delta(X)\}), \tag{2.1}$$

where we put, by definition, $C:=C(\{X\})$ with a set $\{X\}\subset\mathcal{G}$. It is a trivial consequence that for $\mathcal{A}:=U_z(\mathcal{G})$

$$[\Delta(C),\Delta(X_i)]_{\mathcal{A}\otimes\mathcal{A}}=\Delta([C,X_i]_{\mathcal{A}})=0 \tag{2.2}$$

for any $X_i\in\mathcal{G},\ \ i=\overline{1,n}$.

Define now recurrently the following $N-th$ coproduct $\Delta^{(N)}:\mathcal{A}\to\overset{(N+1)}{\otimes}\mathcal{A}$ for any $N\in\mathbb{Z}_+$, where $\Delta^{(2)}:=\Delta$ and $\Delta^{(1)}:=id$ and

$$\Delta^{(N)}:=((id\otimes)^{N-2}\otimes\Delta)\cdot\Delta^{(N-1)}, \tag{2.3}$$

or as

$$\Delta^{(N)} := (\Delta \otimes (id\otimes)^{N-2} \otimes id \otimes id) \cdot \Delta^{(N-1)}. \tag{2.4}$$

One can straightforwardly verify that

$$\Delta^{(N)} := (\Delta^{(m)} \otimes \Delta^{(N-m)}) \cdot \Delta \tag{2.5}$$

for any $m = \overline{0, N}$, and the mapping $\Delta^{(N)} : \mathcal{A} \to \overset{(N+1)}{\otimes} \mathcal{A}$ is an algebras homomorphism, that is

$$[\Delta^{(N)}(X), \Delta^{(N)}(Y)]_{\overset{(N+1)}{\otimes} \mathcal{A}} = \Delta^{(N)}([X,Y]_{\mathcal{A}}) \tag{2.6}$$

for any $X, Y \in \mathcal{A}$. In a particular case if $\mathcal{A}$ $=U(\mathcal{G})$, the following exact expression

$$\begin{aligned} \Delta^{(N)}(X) \quad = \quad & X(\otimes id)^{N-1} \otimes id + id \otimes X(\otimes id)^{N-1} \otimes id + ... \\ & ... + (\otimes id)^{N-1} \otimes id \otimes X \end{aligned} \tag{2.7}$$

holds for any $X \in \mathcal{G}$.

3 Poisson co-algebras and their realizations

As is well known [5], [6], a Poisson algebra $\mathcal{P}$ is a vector space endowed with a commutative multiplication and a Lie bracket $\{.,.\}$ including a derivation on $\mathcal{P}$ in the form

$$\{a, bc\} = b\{a, c\} + \{a, b\}c \tag{3.1}$$

for any a, b and $c \in \mathcal{P}$. If $\mathcal{P}$ and $\mathcal{Q}$ are Poisson algebras one can naturally define the following Poisson structure on $\mathcal{P} \otimes \mathcal{Q}$:

$$\{a \otimes b, c \otimes d\}_{\mathcal{P} \otimes \mathcal{Q}} = \{a, c\}_{\mathcal{P}} \otimes (bd) + (ac) \otimes \{b, d\}_{\mathcal{Q}} \tag{3.2}$$

for any $a, c \in \mathcal{P}$ and $b, d \in \mathcal{Q}$. We shall also say that $(\mathcal{P}\ ; \Delta)$ is a Poisson co-algebra if $\mathcal{P}$ is a Poisson algebra and $\Delta : \mathcal{P} \to \mathcal{P} \otimes \mathcal{P}$ is a Poisson algebras homomorphism, that is

$$\{\Delta(a), \Delta(b)\}_{\mathcal{P} \otimes \mathcal{P}} = \Delta(\{a, b\}_{\mathcal{P}}) \tag{3.3}$$

for any $a, b \in \mathcal{P}$.

It is useful to note here that any Lie algebra $\mathcal{G}$ generates naturally a Poisson co-algebra $(\mathcal{P}; \Delta)$ by defining a Poisson bracket on $\mathcal{P}$ by means of the following expression: for any $a, b \in \mathcal{P}$

$$\{a, b\}_{\mathcal{P}}; =< grad, \vartheta grad b > . \tag{3.4}$$

Here $\mathcal{P} \simeq C^{\infty}(\mathbb{R}^n; \mathbb{R})$ is a space of smooth mappings linked with a base variables of the Lie algebra $\mathcal{G}$, $n = \dim \mathcal{G}$, and the implectic [6] matrix $\vartheta : T^*(\mathcal{P}) \to T(\mathcal{P})$ is given as

$$\vartheta(x) = \{\sum_{k=1}^{n} c_{ij}^k x_k : i, j = \overline{1, n}\}, \tag{3.5}$$

where c_{ij}^k, $i,j,k=\overline{1,n}$, are the corresponding structure constants of the Lie algebra $\mathcal{G}$ and $x \in \mathbb{R}^n$ are the corresponding linked coordinates. It is easy to check that the coproduct (1.2) is a Poisson algebras homomorphism between $\mathcal{P}$ and $\mathcal{P} \otimes \mathcal{P}$. If one can find a "quantum" deformation $U_z(\mathcal{G})$, then the corresponding Poisson co-algebra $\mathcal{P}_z$ can be constructed making use of the naturally deformed implectic matrix $\vartheta_z : T^*(\mathcal{P}_z) \to T(\mathcal{P}_z)$. For instance, if $\mathcal{G} = so(2,1)$, there exists a deformation $U_z(so(2,1))$ defined by the following deformed commutator relations with a parameter $z \in \mathbb{C}$:

$$\begin{aligned}
[\tilde{X}_2, \tilde{X}_1] &= \tilde{X}_3, [\tilde{X}_2, \tilde{X}_3] = -\tilde{X}_1, \\
[\tilde{X}_3, \tilde{X}_1] &= \frac{1}{z}\sinh(z\tilde{X}_2),
\end{aligned} \tag{3.6}$$

where at $z=0$ elements $\tilde{X}_i\Big|_{z=0} = X_i \in so(2,1)$, $i=\overline{1,3}$, compile a base of generators of the Lie algebra $so(2,1)$. Then, based on expressions (3.6) one can easily construct the corresponding Poisson co-algebra $\mathcal{P}_z$, endowed with the implectic matrix

$$\vartheta_z(\tilde{x}) = \begin{pmatrix} 0 & -\tilde{x}_3 & -\frac{1}{z}\sinh(z\tilde{x}_2) \\ \tilde{x}_3 & 0 & -\tilde{x}_1 \\ \frac{1}{z}\sinh(z\tilde{x}_2) & \tilde{x}_1 & 0 \end{pmatrix} \tag{3.7}$$

for any point $\tilde{x} \in \mathbb{R}^3$, linked naturally with the deformed generators $\tilde{X}_i$, $i=\overline{1,3}$, taken above. Since the corresponding coproduct on $U_z(so(2,1))$ acts on this deformed base of generators as

$$\begin{aligned}
\Delta(\tilde{X}_2) &= I \otimes \tilde{X}_2 + \tilde{X}_2 \otimes I, \\
\Delta(\tilde{X}_1) &= \exp(-\frac{z}{2}\tilde{X}_2) \otimes \tilde{X}_1 + \tilde{X}_1 \otimes \exp(\frac{z}{2}\tilde{X}_2), \\
\Delta(\tilde{X}_2) &= \exp(-\frac{z}{2}\tilde{X}_2) \otimes \tilde{X}_3 + \tilde{X}_3 \otimes \exp(\frac{z}{2}\tilde{X}_2),
\end{aligned} \tag{3.8}$$

satisfying the main homomorphism property for the whole deformed universal enveloping algebra $U_z(so(2,1))$.

Consider now some realization of the deformed generators $\tilde{X}_i \in U_z(\mathcal{G})$, $i = \overline{1,n}$, that is a homomorphism mapping $D_z : U_z(\mathcal{G}) \to \mathcal{P}(M)$, such that

$$D_z(\tilde{X}_i) = \tilde{e}_i, \tag{3.9}$$

$i=\overline{1,n}$, are some elements of a Poisson manifold $\mathcal{P}(M)$ realized as a space of functions on a finite-dimensional manifold M, satisfying the deformed commutator relationships

$$\{\tilde{e}_i, \tilde{e}_j\}_{\mathcal{P}(M)} = \vartheta_{z,ij}(\tilde{e}), \tag{3.10}$$

where, by definition, expressions $[\tilde{X}_i, \tilde{X}_j] = \vartheta_{z,ij}(\tilde{X})$, $i,j=\overline{1,n}$, generate a Poisson co-algebra structure on the function space $\mathcal{P}_z := \mathcal{P}_z(\mathcal{G})$ linked with a given Lie algebra $\mathcal{G}$. Making use of the homomorphism property (3.3) for the

coproduct mapping $\Delta : \mathcal{P}_z(\mathcal{G}) \to \mathcal{P}_z(\mathcal{G}) \otimes \mathcal{P}_z(\mathcal{G})$, one finds that for all $i, j = \overline{1, n}$

$$\{\Delta(\tilde{x}_i), \Delta(\tilde{x}_j)\}_{\mathcal{P}_z(\mathcal{G}) \otimes \mathcal{P}_z(\mathcal{G})} = \Delta(\{\tilde{x}_i, \tilde{x}_j\}_{\mathcal{P}_z(\mathcal{G})} = \vartheta_{z,ij}(\Delta(\tilde{x})) \tag{3.11}$$

and for the corresponding coproduct $\Delta : \mathcal{P}(M) \to \mathcal{P}(M) \otimes \mathcal{P}(M)$ one gets similarly

$$\{\Delta(\tilde{e}_i), \Delta(\tilde{e}_j)\}_{\mathcal{P}(M) \otimes \mathcal{P}(M)} = \Delta(\{\tilde{e}_i, \tilde{e}_j\}_{\mathcal{P}(M)} = \vartheta_{z,ij}(\Delta(\tilde{e})), \tag{3.12}$$

where $\{.,.\}_{\mathcal{P}(M)}$ is some, eventually, canonical Poisson structure on a finite-dimensional manifold M.

Let $q \in M$ be a point of M and consider its coordinates as elements of $\mathcal{P}(M)$. Then one can define the following elements

$$q_j := (I\otimes)^{j-1} q (\otimes I)^{N-j} \in \overset{(N)}{\otimes} \mathcal{P}(M), \tag{3.13}$$

where $j = \overline{1, N}$ by means of which one can construct the corresponding N-tuple realization of the Poisson co-algebra structure (3.12) as follows:

$$\{\tilde{e}_i^{(N)}, \tilde{e}_j^{(N)}\}_{\overset{(N)}{\otimes} \mathcal{P}(M)} = \vartheta_{z,ij}(\tilde{e}^{(N)}), \tag{3.14}$$

with $i, j = \overline{1, n}$ and

$$\overset{(N)}{\otimes} D_z(\Delta^{(N-1)}(\tilde{e}_i) := \tilde{e}_i^{(N)}(q_1, q_2, ..., q_N). \tag{3.15}$$

For instance, for the $U_z(so(2,1))$ case (3.6), one can take [7] the realization Poisson manifold $\mathcal{P}(M) = \mathcal{P}(\mathbb{R}^2)$ with the standard canonical Heisenberg-Weil Poissonian structure on it:

$$\{q, q\}_{\mathcal{P}(\mathbb{R}^2)} = 0 = \{p, p\}_{\mathcal{P}(\mathbb{R}^2)}, \qquad \{p, q\}_{\mathcal{P}(\mathbb{R}^2)} = 1, \tag{3.16}$$

where $(q, p) \in \mathbb{R}^2$. Then expressions (3.15) for $N = 2$ give rise to the following relationships

$$: \begin{array}{ccc} \tilde{e}_1^{(2)}(q_1, q_2, p_1, p_2) & := & (D_z \otimes D_z)\Delta(\tilde{X}_1) = \\ 2\frac{\sinh(\frac{z}{2}p_1)}{z} \cos q_1 \exp(\frac{z}{2}p_1) & + & 2\exp(-\frac{z}{2}p_1))\frac{\sinh(\frac{z}{2}p_2)}{z} \cos q_2, \\ \tilde{e}_2^{(2)}(q_1, q_2, p_1, p_2) & := & (D_z \otimes D_z)\Delta(\tilde{X}_2) = p_1 + p_2, \\ \tilde{e}_3^{(2)}(q_1, q_2, p_1, p_2) & := & (D_z \otimes D_z)\Delta(\tilde{X}_3) = \\ 2\frac{\sinh(\frac{z}{2}p_1)}{z} \sin q_1 \exp(\frac{z}{2}p_2) & + & 2\exp(-\frac{z}{2}p_1))\frac{\sinh(\frac{z}{2}p_2)}{z} \sin q_2, \end{array} \tag{3.17}$$

where elements $(q_1, q_2, p_1, p_2) \in \mathcal{P}(\mathbb{R}^2) \otimes \mathcal{P}(\mathbb{R}^2)$ satisfy the induced by (3.16) Heisenberg-Weil commutator relations:

$$\{q_i, q_j\}_{\mathcal{P}(\mathbb{R}^2)\otimes\mathcal{P}(\mathbb{R}^2)} = 0 = \{p_i, p_j\}_{\mathcal{P}(\mathbb{R}^2)\otimes\mathcal{P}(\mathbb{R}^2)}, \quad \{p_i, q_j\}_{\mathcal{P}(\mathbb{R}^2)\otimes\mathcal{P}(\mathbb{R}^2)} = \delta_{ij} \tag{3.18}$$

for any $i, j = \overline{1, 2}$.

4 Casimir elements and the Heisenberg-Weil algebra related algebraic structures

Consider any Casimir element $\tilde{C} \in U_z(\mathcal{G})$ related with an $\mathbb{R} \ni z-$deformed Lie algebra $\mathcal{G}$ structure in the form

$$[\tilde{X}_i, \tilde{X}_j] = \vartheta_{z,ij}(\tilde{X}), \tag{4.1}$$

where $i, j = \overline{1,n}$, $n = \dim \mathcal{G}$, and, by definition, $[\tilde{C}, \tilde{X}_i] = 0$. The following general lemma holds.

Lemma 1 *Let $(U_z(\mathcal{G}); \Delta)$ be a co-algebra with generators satisfying (4.1) and $\tilde{C} \in U_z(\mathcal{G})$ be its Casimir element; then*

$$[\Delta^{(m)}(\tilde{C}), \Delta^{(N)}(\tilde{X}_i)]_{\overset{(N+1)}{\otimes} U_z(\mathcal{G})} = 0 \tag{4.2}$$

for any $i = \overline{1,n}$ and $m = \overline{1,N}$.

As a simple corollary of this Lemma one finds from (4.2) that

$$[\Delta^{(m)}(\tilde{C}), \Delta^{(N)}(\tilde{C})]_{\overset{(N+1)}{\otimes} U_z(\mathcal{G})} = 0$$

for any $k, m \in \mathbb{Z}_+$.

Consider now some realization (3.9) of our deformed Poisson co-algebra structure (4.1) and check that the expression

$$[\Delta^{(m)}(C(\tilde{e}), \Delta^{(N)}(\mathcal{H}(\tilde{e}))]_{\overset{(N+1)}{\otimes} \mathcal{P}(M)} = 0 \tag{4.3}$$

too for any $m = \overline{1,N}$, $N \in \mathbb{Z}_+$, if $C(\tilde{e}) \in I(\mathcal{P}(M))$, that is $\{C(\tilde{e}), q\}_{\mathcal{P}(M)} = 0$ for any $q \in M$. Since

$$\mathcal{H}^{(N)}(q) := \Delta^{^{(N-1)}}(\mathcal{H}(\tilde{e})) \tag{4.4}$$

are in general, smooth functions on $\overset{(N+1)}{\otimes} M$, which can be used as Hamilton ones subject to the Poisson structure on $\overset{(N+1)}{\otimes} \mathcal{P}(M)$, the expressions (4.4) mean nothing else that functions

$$\gamma^{(m)}(q) := \Delta^{^{(N)}}(C(\tilde{e})) \tag{4.5}$$

are their invariants, that is

$$\{\gamma^{(m)}(q), \mathcal{H}^{^{(N)}}(q)\}_{\overset{(N+1)}{\otimes} \mathcal{P}(M)} = 0 \tag{4.6}$$

for any $m = \overline{1,N}$. Thereby, the functions (4.4) and (4.5) generate under some additional but natural conditions a hierarchy of a priori Liouville-Arnold integrable Hamiltonian flows on the Poisson manifold $\overset{(N+1)}{\otimes} \mathcal{P}(M)$.

Consider now a case when a Poisson manifold $\mathcal{P}(M)$ and its co-algebra deformation $\mathcal{P}_z(\mathcal{G})$. Thus for any coordinate points $x_i \in \mathcal{P}(\mathcal{G})$, $i = \overline{1,n}$, the following relationships

$$\{x_i, x_j\} = \sum_{k=1}^{n} c_{ij}^k x_k := \vartheta_{ij}(x) \tag{4.7}$$

define a Poisson structure on $\mathcal{P}(\mathcal{G})$, related with the corresponding Lie algebra structure of $\mathcal{G}$, and there exists a representation (3.9), such that elements $\tilde{e}_i := D_z(\tilde{X}_i) = \tilde{e}_i(x)$ satisfy the relationships $\{\tilde{e}_i, \tilde{e}_j\}_{\mathcal{P}_z(\mathcal{G})} = \vartheta_{z,ij}(\tilde{e})$ for any $i = \overline{1,n}$, with the limiting conditions

$$\lim_{z\to 0} \vartheta_{z,ij}(\tilde{e}) = \sum_{k=1}^{n} c_{ij}^k x_k, \quad \lim_{z\to 0} \tilde{e}_i(x) = x_i \tag{4.8}$$

for any $i, j = \overline{1,n}$ being held. For instance, take the Poisson co-algebra $\mathcal{P}_z(so(2,1))$ for which there exists a realization (3.9) in the following form:

$$\begin{aligned} \tilde{e}_1 \quad &: \; = D_z(\tilde{X}_1) = \frac{\sinh(\frac{z}{2}x_2)}{zx_2} x_1, \;\; \tilde{e}_2 := D_z(\tilde{X}_2) = x_2, \\ \tilde{e}_3 \quad &: \; = D_z(\tilde{X}_3) = \frac{\sinh(\frac{z}{2}x_2)}{zx_2} x_3, \end{aligned} \tag{4.9}$$

where $x_i \in \mathcal{P}(so(2,1))$, $i = \overline{1,3}$, satisfy the $so(2,1)-$commutator relationships

$$\begin{aligned} \{x_2, x_1\}_{\mathcal{P}(so(2,1))} &= x_3, \;\; \{x_2, x_3\}_{\mathcal{P}(so(2,1))} = -x_1, \\ \{x_3, x_1\}_{\mathcal{P}(so(2,1))} &= x_{2,} \end{aligned} \tag{4.10}$$

with the coproduct operator $\Delta : \mathcal{U}_z(so(2,1)) \to \mathcal{U}_z(so(2,1)) \otimes \mathcal{U}_z(so(2,1))$ being

given by (3.8). It is easy to check that conditions (4.7) and (4.8) hold.

The next example is related with the co-algebra $\mathcal{U}_z(\pi(1,1))$ of the Poincare algebra $\pi(1,1)$ for which the following non-deformed relationships

$$[X_1, X_2] = X_3, \;\; [X_1, X_3] = X_2, \;\; [X_3, X_2] = 0 \tag{4.11}$$

hold. The corresponding coproduct $\Delta : \mathcal{U}_z(\pi(1,1)) \to \mathcal{U}_z(\pi(1,1)) \otimes \mathcal{U}_z(\pi(1,1))$ is given by the Woronowicz [8] expressions

$$\begin{aligned} \Delta(\tilde{X}_1) &= I \otimes \tilde{X}_1 + \tilde{X}_1 \otimes I, \\ \Delta(\tilde{X}_2) &= \exp(-\frac{z}{2}\tilde{X}_1) \otimes \tilde{X}_1 + \tilde{X}_1 \otimes \exp(\frac{z}{2}\tilde{X}_1), \\ \Delta(\tilde{X}_3) &= \exp(-\frac{z}{2}\tilde{X}_1) \otimes \tilde{X}_3 + \tilde{X}_3 \otimes \exp(\frac{z}{2}\tilde{X}_1), \end{aligned} \tag{4.12}$$

where $z \in \mathbb{R}$ is a prarameter. Under the deformed expressions (4.12) the elements $\tilde{X}_j \in \mathcal{U}_z(\pi(1,1))$, $j = \overline{1,3}$,satisfy still undeformed commutator relationships, that is $\vartheta_{z,ij}(\tilde{X}) = \vartheta_{ij}(X)|_{X \Rightarrow \tilde{X}}$ for any $z \in \mathbb{R}$, $i, j = \overline{1,3}$, being given by

(4.11). As a result, we can state that $\tilde{e}_i := D_z(\tilde{X}_i) = \tilde{e}_i(x) = x_i$, where for $x_i \in \mathcal{P}(\pi(1,1))$, $i = \overline{1,3}$, the following Poisson structure

$$\begin{aligned} \{x_1, x_2\}_{\mathcal{P}(\pi(1,1))} &= x_3, \quad \{x_1, x_3\}_{\mathcal{P}(\pi(1,1))} = x_2, \\ \{x_3, x_2\}_{\mathcal{P}(\pi(1,1))} &= 0 \end{aligned} \tag{4.13}$$

holds. Moreover, since $C = x_2^2 - x_3^2 \in I(\mathcal{P}(\pi(1,1)))$, that is $\{C, x_i\}_{\mathcal{P}(\pi(1,1))} = 0$ for any $i = \overline{1,3}$, on can construct, making use of (4.4) and (4.5), integrable Hamiltonian systems on $\overset{(N)}{\otimes}\mathcal{P}(\pi(1,1))$.The same one can do subject to the discussed above Poisson co-algebra $\mathcal{P}_z(so(2,1))$ realized by means of the Poisson manifold $\mathcal{P}(so(2,1))$, taking into account that the following element $C = x_2^2 - x_1^2 - x_3^2 \in I(\mathcal{P}(so(2,1)))$ is a Casimir one.

Now we will consider a special extended Heisenberg-Weil co-algebra $\mathcal{U}_z(h_4)$, called still the oscillator co-algebra. The undeformed Lie algebra h_4 commutator relationships take the form:

$$\begin{aligned} [n, a_+] &= a_+, \quad [n, a_-] = -a_-, \\ [a_-, a_+] &= m, \quad [m, \cdot] = 0, \end{aligned} \tag{4.14}$$

where $\{n, a_\pm, m\} \subset h_4$ compile a basis of $h_4, \dim h_4 = 4$. The Poisson co-algebra $\mathcal{P}(h_4)$ is naturally endowed with the Poisson structure like (4.14) and admits its realization (3.9) on the Poisson manifold $\mathcal{P}(\mathbb{R}^2)$. Namely, on $\mathcal{P}(\mathbb{R}^2)$ one has

$$\begin{aligned} e_\pm &= D(a_\pm) = \sqrt{p}\exp(\mp q), \\ e_1 &= D(m) = 1, \ e_0 = D(n) = p, \end{aligned} \tag{4.15}$$

where $(q,p) \in \mathbb{R}^2$ and the Poisson structure on $\mathcal{P}(\mathbb{R}^2)$ is canonical, that is the same as (3.16).

Closely related with the relationships (4.14) there is a generalized $\mathcal{U}_z(su(2))$ co-algebra, for which

$$\begin{aligned} [x_3, x_\pm] &= \pm x_\pm, \qquad [y_\pm, \cdot] = 0, \\ [x_+, x_-] &= y_+ \sin(2zx_3) + y_- \cos(2zx_3))\frac{1}{\sin z}, \end{aligned} \tag{4.16}$$

where $z \in \mathbb{C}$ is an arbitrary parameter. The co-algebra structure is given now as follows:

$$\begin{aligned} \Delta(x_\pm) &= c^\pm_{1(2)} e^{izx_3} \otimes x_\pm + x_\pm \otimes c^\pm_{2(1)} e^{-izx_3}, \\ \Delta(x_3) &= I \otimes x_3 + x_3 \otimes I, \ \Delta(c_i^\pm) = c_i^\pm \otimes c_i^\pm, \\ \nu(x_\mp) &= -(c^\pm_{1(2)})^{-1} e^{-izx_3} x_\mp e^{izx_3} (c^\pm_{2(1)})^{-1}, \\ \nu(c_i^\pm) &= (c_i^\pm)^{-1}, \ \nu(e^{\pm izx_3}) = e^{\mp izx_3} \end{aligned} \tag{4.17}$$

with $c_i^{\pm} \in \mathcal{U}_z(su(2))$, $i = \overline{1,2}$, being fixed elements. One can check that the corresponding to (4.16) Poisson structure on $\mathcal{P}_z(su(2))$ can be realized by means of the canonical Poisson structure on the phase space $\mathcal{P}(\mathbb{R}^2)$ as follows:

$$\begin{aligned} [q,p] &= i, \qquad D_z(x_3) = q, \qquad D_z(x_{\mp}) = e^{\pm ip} g_z(q), \\ g_z(q) &= (k + \sin[z(s-q)])(y_+ \sin[(q+s+1)] + y_- \cos[z(q+s+1)])^{1/2} \frac{1}{\sin z}, \end{aligned} \tag{4.18}$$

where $k, s \in \mathbb{C}$ are constant parameters. Thereby making use of (4.5) and (4.6), one can construct a new class of Liouville integrable Hamiltonian flows.

5 The Heisenberg-Weil co-algebra structure and related integrable flows

Consider the Heisenberg-Weil algebra commutator relationships (4.14) and related with them the fo9llowing homogenous quadratic forms

$$\left.\begin{array}{c} x_1x_2 - x_2x_1 - \alpha x_3^2 = 0, \\ x_1x_3 - x_3x_1 = 0, \quad x_2x_3 - x_3x_2 = 0 \end{array}\right\} R(x), \tag{5.1}$$

where $\alpha \in \mathbb{C}$, $\quad x_i \in A$, $i = \overline{1,3}$, are some elements of a free associative algebra A. The quadratic algebra $A/R(x)$ can be deformed via

$$\left.\begin{array}{c} x_1x_2 - z_1x_2x_1 - \alpha x_3^2 = 0, \\ x_1x_3 - z_2x_3x_1 = 0, \quad x_2x_3 - z_2^{-1}x_3x_2 = 0, \end{array}\right\} R_z(x), \tag{5.2}$$

where $z_1, z_2 \in \mathbb{C}\backslash\{0\}$ are some parameters.

Let V be the vector space of columns $X := (x_1, x_2, x_3)^{\intercal}$ and define the following action

$$h_T : V \to (V \otimes V^*) \otimes V, \tag{5.3}$$

where, by definition, for any $X \in V$

$$h_T(X) = T \otimes X. \tag{5.4}$$

It is easy to check that conditions (5.2) will be satisfied if the following relations [9]

$$\begin{aligned} T_{11}T_{33} &= T_{33}T_{11}, \;\; T_{12}T_{33} = z_2^{-2}T_{33}T_{12}, \; T_{21}T_{33} = z_1^2T_{33}T_{21}, \\ T_{22}T_{33} &= T_{33}T_{22}, \;\; T_{31}T_{33} = z_2T_{33}T_{31}, \quad T_{32}T_{33} = z_1^{-1}T_{33}T_{32}, \\ T_{11}T_{12} &= z_1T_{12}T_{11}, \; T_{21}T_{22} = z_1T_{22}T_{21}, \; z_2T_{11}T_{32} - z_2T_{32}T_{11} = \\ &= z_1z_2T_{12}T_{31} - T_{31}T_{12}, \; T_{21}T_{32} - z_1z_2T_{32}T_{21} = \\ &= z_1T_{22}T_{31} - z_2T_{31}T_{22}, \;\; T_{11}T_{22} - T_{22}T_{11} = \\ &= z_1T_{12}T_{21} - z_1^{-1}T_{21}T_{12}, \;\; (T_{11}T_{22} - z_1T_{12}T_{21}) = \\ &= \alpha T_{33}^2 - T_{31}T_{32} + z_1T_{32}T_{31} \end{aligned} \tag{5.5}$$

hold. Put now for further convenience $z_1 = z_2^2 := z^2 \in \mathbb{C}$ and compute the "quantum" determinant $D(T)$ of the matrix $T : (A/R_z(x))^3 \rightarrow (A/R_z(x))^3$:

$$D(T) = (T_{11}T_{22} - z^{-2}T_{21}T_{12})T_{33}. \tag{5.6}$$

Remark here that the determinant (5.6) is not central, that is

$$\begin{array}{rcl} D^{-1}T_{11} & = & T_{11}D^{-1}, \;\; D^{-1}T_{12} = z^{-6}T_{12}D^{-1}, \\ D^{-1}T_{33} & = & T_{33}D^{-1}, \;\; z^{-6}D^{-1}T_{21} = T_{12}D^{-1}, \\ D^{-1}T_{22} & = & T_{22}D^{-1}, \;\; z^{-3}D^{-1}T_{31} = T_{31}D^{-1}, \\ D^{-1}T_{32} & = & z^{-3}T_{32}D^{-1}. \end{array} \tag{5.7}$$

Taking into account properties (5.5) - (5.7), one can construct the Heisenberg-Weil related co-algebra $\mathcal{U}_z(h)$ being a Hopf algebra with the fo9llowing coproduct Δ, counit ε and antipode ν :

$$\begin{array}{rcl} \Delta(T) & : & = T \otimes T, \;\; \Delta(D^{-1}) := D^{-1} \otimes D^{-1}, \\ \varepsilon(T) & : & = I, \;\; \varepsilon(D^{-1}) := I, \;\; \nu(T) := T^{-1}, \; \nu(D) := D^{-1}. \end{array} \tag{5.8}$$

Based now on relationships (5.5), one can easily construct the Poisson tensor

$$\{\Delta(\tilde{T}), \Delta(\tilde{T})\}_{\mathcal{P}_z(h)\otimes\mathcal{P}_z(h)} = \Delta(\{\tilde{T}, \tilde{T}\}_{\mathcal{P}_z(h)}) := \vartheta_z(\Delta(\tilde{T})), \tag{5.9}$$

subject to which all of functionals (4.5) will be commuting to each other, and moreover, will be Casimir ones. Choosing some appropriate Hamiltonian functions $\mathcal{H}^{(N)}(\tilde{T}) := \Delta^{(N-1)}(\mathcal{H}(\tilde{T}))$ for $N \in \mathbb{Z}_+$ one makes it possible to present a priori nontrivial integrable Hamiltonian systems. On the other handside, the co-algebra $\mathcal{U}_z(h)$ built by (5.7) and (5.8) possesses the following fundamental $\mathcal{R}$-matrix [4] property:

$$\mathcal{R}(z)(T \otimes I)(I \otimes T) = (I \otimes T)(T \otimes I)\mathcal{R}(z) \tag{5.10}$$

for some complex-valued matrix $\mathcal{R}(z) \in Aut(\mathbb{C}^3 \otimes \mathbb{C}^3)$, $z \in \mathbb{C}$. The latter, as is well known [4], gives rise to a regular procedure of constructing an infinite hierarchy of Liouville-integrable operator (quantum) Hamiltonian systems on related quantum Poissonian phase spaces. On their special cases interesting for applications we plan to go on in another place.

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